

# Space-efficient Algorithms for Visibility Problems in Simple Polygon

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**Abstract.** Given a simple polygon  $P$  consisting of  $n$  vertices, we study the problem of designing space-efficient algorithms for computing (i) the visibility polygon of a point inside  $P$ , (ii) the weak visibility polygon of a line segment inside  $P$  and (iii) the minimum link path between a pair of points inside  $P$ . For problem (i) two algorithms are proposed. The first one is an in-place algorithm where the input array may be lost. It uses only  $O(1)$  extra space apart from the input array. The second one assumes that the input is given in a read-only array, and it needs  $O(\sqrt{n})$  extra space. The time complexity of both the algorithms are  $O(n)$ . For problem (ii), we have assumed that the input polygon is given in a read-only array. Our proposed algorithm runs in  $O(n^2)$  time using  $O(1)$  extra space. For problem (iii) the time and space complexities of our proposed algorithm are  $O(kn)$  and  $O(1)$  respectively;  $k$  is the length (number of links) in a minimum link path between the given pair of points.

## 1 Introduction

Visibility is one of the foundational areas in computational geometry and it has applications in various domains, including robot motion planning, guarding art galleries, computer graphics, GIS, sensor network. For an illustrated survey, see [4]. Recently, visibility algorithms are being embedded in the hardware of digital cameras, sensors, etc, and the constraint on the size of the instrument has become important. So, the algorithm designers are now becoming interested in developing space-efficient algorithms for various visibility problems.

So far, in-place algorithms have been studied for a very few problems in computational geometry (see [3]). In [1], constant work space algorithms for the following visibility related problems are studied: (i) triangulation of a simple polygon, (ii) triangulation of a point set, (iii) Euclidean shortest path between a pair of points inside a simple polygon, and (iv) Euclidean minimum spanning tree, where the input is given in a read-only array. The time complexity of the algorithms for the problems (i)-(iii) are  $O(n^2)$ , and that for problem (iv) is  $O(n^3)$ . The open question was whether one can compute the visibility of a point inside a simple polygon in sub-quadratic time, where the polygon is given in a read-only array [1]. Recently, two algorithms for this problem are proposed by Barba et al. [2]. The first one is deterministic, and it requires  $O(n\bar{r})$  time and  $O(1)$  space, where  $\bar{r}$  is the number of reflex vertices of the output visibility polygon. The second one is a randomized algorithm and it requires  $O(n \log r)$  time and  $O(\log r)$  space, where  $r$  is the number of reflex vertices in the input polygon.

**New Results:** In this paper we present the following results:

- An in-place algorithm for computing the visibility polygon of a point inside a simple polygon  $P$  in  $O(n)$  time and  $O(1)$  extra work-space, where  $n$  is the number of vertices of  $P$ . Note that, after the execution of the algorithm, the polygon can not be retrieved from the content of the array.
- An  $O(\sqrt{n})$  space algorithm for computing the visibility polygon of a point inside the polygon  $P$  in  $O(n)$  time, where the vertices of  $P$  are given in a read-only array.
- An  $O(n^2)$  time and  $O(1)$  extra work-space algorithm for computing the weak-visibility of the polygon  $P$  from a line segment inside the polygon, where the vertices of  $P$  are given in a read-only array.
- An  $O(kn)$  time and  $O(1)$  extra work-space algorithm for computing a minimum link path between a pair of points  $s$  and  $t$  inside  $P$ , where the vertices of  $P$  are given in a read-only array and  $k$  is the size of the output.

## 2 Visibility of a point inside simple polygon

Let the vertices of the polygon  $P = \{p_1, p_2, \dots, p_n\}$  be given in an array  $P$  in anticlockwise order. Initially,  $P[i]$  contains the coordinates of the  $i$ -th vertex  $p_i$  of  $P$  in the given order. Let  $\pi$  be a given point inside  $P$ . We will consider the problem of computing the visibility polygon of  $\pi$  in  $P$ . It is easy to see that the visibility polygon of  $\pi$  can be computed in  $O(n^2)$  time and  $O(1)$  extra space where the vertices of  $P$  are given in a read-only array. We will describe two algorithms, namely INPLACE\_VISIBILITY and READONLY\_VISIBILITY. The first one computes the visibility polygon in in-place manner with  $O(1)$  extra space. The second one assumes that the input array is read-only, and it uses  $O(\sqrt{n})$  extra space. The time complexity of both the algorithms are  $O(n)$ .

### 2.1 INPLACE\_VISIBILITY

We first describe an algorithm to report all the vertices of  $P$  that are visible from  $\pi$ . Later, we show that the above algorithm can be easily modified to report the visibility polygon of  $\pi$ . We start the algorithm by drawing a horizontal ray  $\vec{H}$  through the point  $\pi$  to its right that finds an edge  $e_\theta = (p_\theta, p_{\theta+1})$  of  $P$  intersected by  $\vec{H}$  first. Let  $q$  be the point of intersection  $\vec{H}$  and  $e_\theta$  (see Figure 1(a)).

We visit the vertices of  $P$  in counterclockwise order starting from  $p_{\theta+1}$ . After visiting all the vertices of  $P$ , the visibility polygon of  $\pi$  will be stored in consecutive locations of the input array. Note that, at some point of time during the execution, a vertex may be visible to  $\pi$ , but as the algorithm proceeds, it may not remain visible.

**Observation 1** *Let  $p_i$  and  $p_j$  be a pair of visible vertices at an instant of time, where  $(i - \theta) \bmod n < (j - \theta) \bmod n$ . If both  $p_i$  and  $p_j$  become invisible during the further execution of the algorithm, then  $p_j$  becomes invisible prior to  $p_i$ .*

As in the classical algorithm for computing the visibility polygon [4], here also we store the vertices of  $P$  visible to  $\pi$  in a stack. At the end of the execution, the content of the stack indicates the vertices of the visibility polygon of  $\pi$ . Due to

the constraint on space, here we maintain the stack in the array  $P$  itself. We use three index variables  $i$ ,  $k$  and  $\ell$ , where  $k$  and  $\ell$  denote respectively the starting and ending indices of the stack at an instant of time, and  $i$  indicates the index of the current vertex of  $P$  under processing. We use two more workspaces  $\Phi$  and  $\Psi$  that stores  $\angle q\pi P[\ell]$  and  $\angle q\pi P[i]$  respectively, where  $\ell$  and  $i$  are as stated above. Note that the content of  $\Phi$  can be from  $0^\circ$  to  $360^\circ$ . But  $\Psi$  can contain negative angle. This happens when the traversal path along the boundary of the polygon crosses the ray  $\vec{H} = \pi q$  an even number of times.

The execution starts from the vertex  $P[\theta + 1]$ . Initially we set  $k = \ell = \theta + 1$ ,  $i = \theta + 2$ ,  $\Phi = \Psi = 0$ . We process each vertex  $P[i]$ ,  $i = \theta + 2, \theta + 3, \dots, n, 1, 2, \dots, \theta$  in this order. While processing  $P[i]$ , we compute  $\Psi = \angle q\pi P[i]$  as follows:

- If  $P[i - 1] \rightarrow P[i]$  is an anticlockwise turn then  $\Psi = \Psi + \angle P[i - 1]\pi P[i]$ ;
- Otherwise  $\Psi = \Psi - \angle P[i - 1]\pi P[i]$ .

Next, we compare  $\Phi$  with  $\Psi$ . Here one of the following three cases may arise. The appropriate actions in each case is explained below. At the end of the execution, the visibility polygon is available in  $P[k \dots \ell]$ .

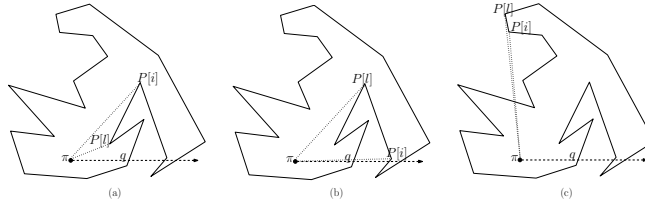


Fig. 1: Processing of vertex  $P[i]$

**Case 1:**  $\Phi \leq \Psi$  (see Figure 1(a)).

**Case 2:**  $\Phi > \Psi$  and the polygon makes a right turn at  $P[i]$  (see Figure 1(b)).

**Case 3:**  $\Phi > \Psi$  and the polygon makes a left turn at  $P[i]$  (see Figure 1(c)).

In Case 1,  $P[i]$  is visible from the point  $\pi$ . We do the following: (i) push  $P[i]$  in the stack by setting  $\ell = (\ell + 1) \bmod n$  and placing  $P[i]$  in location  $P[\ell]$ , (ii) set  $\Phi = \Psi$ , and (iii) increment  $i$  by setting  $i = (i + 1) \bmod n$  to process next vertex. In Case 2, we ignore  $P[i]$  and increment  $i$  (by setting  $i = (i + 1) \bmod n$ ) until it encounters some vertex satisfying Case 1 (see Figure 2(a)).

In Case 3, we pop elements from the stack by (i) setting  $\Phi = \Phi - \angle P[\ell]\pi P[\ell - 1]$  and (ii) decreasing  $\ell$  by 1 at each step until one of the followings hold:

**Case 3.1:**  $\ell$  becomes less than  $k$ , i.e., the stack becomes empty. Here  $k$  and  $\ell$  are reset to  $i$  (see Figure 2(b)).

**Case 3.2:**  $P[\ell]$  and the current  $P[i]$  satisfy  $\Phi < \Psi$  (see Figure 2(c)). Now, process  $P[i]$  as in Case 1.

**Case 3.3:**  $P[\ell]$  and  $P[i]$  satisfy  $\Phi > \Psi$  and the line segments  $[P[i - 1], P[i]]$  and  $[P[\ell], P[\ell + 1]]$  intersect in their interior (see Figure 2(d)). Here we need to proceed (by incrementing  $i$  by 1 at each step) until a vertex  $P[i']$  is obtained that satisfies Case 1. Now we process  $P[i']$  as in Case 1.

Note that, in order to check Case 3.3, we need  $P[i-1]$  and  $P[\ell+1]$ ;  $P[i-1]$  may be lost during the execution of the algorithm. So we need to maintain  $P[i-1]$  in a scalar location *previous\_vertex*.  $P[\ell+1]$  is the last deleted vertex from the stack; it is available since no other vertex is inserted yet in the stack.

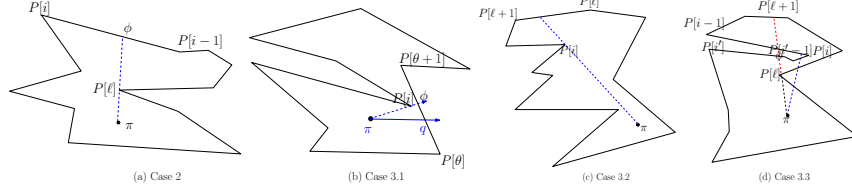


Fig. 2: Different cases that arise during the execution of our in-place algorithm

### Necessary modification required to obtain the entire visibility polygon

In order to compute the entire visibility polygon, we need to modify the execution steps of Case 2 and Case 3 (where  $P[i]$  is not visible from  $\pi$ ) as follows:

In Case 2, after computing the vertex  $i$  satisfying Case 1, we create a vertex  $\phi$  at the point of intersection of the edge  $(p_{i-1}, p_i)$  and the line joining  $\pi$  and  $P[\ell]$  (see Figure 2(a)). The point  $\phi$  is pushed in the stack. Next, the vertex  $p_i$  is processed as in Case 1.

In Case 3, while backtracking (popping vertices from stack) by decrementing  $\ell$  by 1 at each step, we may arrive at one of the above three situations.

In Case 3.1,  $q$  is a vertex of the visibility polygon. Thus, apart from resetting  $k$  and  $\ell$ , we need to insert  $q$  in the stack (see Figure 2(b)). Note that, during the entire execution, this case may appear at most once.

In Case 3.2, a new vertex  $\phi$  is created at the point of intersection of the line joining  $[\pi, P[i])$  and the edge  $(P[\ell], P[\ell+1])$  of  $P$  (see Figure 2(c)). Next,  $\phi$  and  $P[i]$  are inserted in the stack.

In Case 3.3, after computing  $P[i']$ , a new vertex  $\phi$  is created at the point of intersection of the line joining  $(\pi, P[\ell])$  and the edge  $(P[i'-1], P[i'])$  of  $P$  (see Figure 2(d)). Next,  $\phi$  and  $P[i']$  are inserted in the stack.

The pseudo-code of Algorithm 1 is given in the appendix. The correctness of the algorithm follows from Lemmas 3-7 of [5] and Lemmas 1 stated below.

**Lemma 1.** *The inplace maintenance of the stack in the same array  $P$  does not erase the polygonal vertices prior to its processing.*

*Proof.* Consider a *push* operation for maintaining the visibility polygonal vertices during the execution. Such a vertex may be either (i) a vertex of  $P$  or (ii) a point on an edge of  $P$ . When a polygonal vertex is pushed in the stack (Case (i)),  $i$  is immediately incremented. Thus, it does not erase any unprocessed polygonal vertex. In Case (ii), we need to mention that each pair of consecutive edges of the stack defines an edge of the visibility polygon [5]. While processing  $P[i]$ , if a point  $\phi$  on the visibility polygonal edge  $(\alpha, \beta)$  is created, which is not a polygonal vertex, then both  $\alpha$  and  $\beta$  are present in the stack. We pop  $\alpha$  and push  $\phi$  in the stack. Thus the stack does not overlap unprocessed vertices.  $\square$

Apart from the array  $P$ , we need at most a constant number of scalar locations to run the algorithm. Its time complexity depends on the number of changes in the value of  $i$  and  $\ell$ . Since  $i$  is never decremented,  $i$  is modified  $n$  times.  $\ell$  is also modified  $O(n)$  times since it is incremented at most  $n$  times, and the number of decrements of  $\ell$  is bounded by the number of its increments. If there is any reset in  $k$  during the execution, then  $\ell$  is also changed. Thus changes in  $k$  does not affect the complexity of the algorithm. Thus, we have the following result:

**Theorem 1.** *The time complexity of the algorithm INPLACE\_VISIBILITY is  $O(n)$ , and it uses  $O(1)$  extra work-space.*

## 2.2 READONLY\_VISIBILITY

Here, we assume that the vertices of the input polygon are given in a read-only array  $P$  in counterclockwise order, and show that the visibility polygon for a point  $\pi$  inside  $P$  can be computed in  $O(n)$  time using  $O(\sqrt{n})$  extra spaces.

As in the earlier section, here also we draw a horizontal ray  $\vec{H}$  from the point  $\pi$  towards right that meets the edge  $e_\theta = (p_\theta, p_{\theta+1})$ . We first partition the polygon into  $\lceil \sqrt{n} \rceil$  polygonal chains (polychains); each containing  $\sqrt{n}$  vertices except the last one, which may contain fewer number of vertices. The first vertex of the first chain is  $p_{\theta+1}$ . The  $j$ -th polychain will be referred to as  $P_j$ . We start processing from  $p_{\theta+1} \in P_1$ , and process the polychains in counterclockwise order. While processing  $P_j$ , the vertices of  $P_j$  are also processed in counterclockwise manner.

**Definition 1.** *A polychain  $P_j$  is called processed if all its vertices are processed.* We maintain two arrays of integers:  $S$  of size  $\lfloor \sqrt{n} \rfloor$ , and  $R$  of size  $2 \times \lceil \sqrt{n} \rceil$ .

Let us assume that we have processed the chains  $P_1, \dots, P_{j-1}$  and next we want to process  $P_j$ . The array  $S$  contains the indices of the vertices in  $P_j$ . At an instant of time, if any part of the polychain  $P_j$  is visible from  $\pi$ , then  $R[1, j]$  and  $R[2, j]$  stores the indices of two vertices of  $P$  that blocks the visibility of  $P_j$  from its left and right sides at that instant of time. If the first (resp. last) vertex of  $P_j$  is visible to  $\pi$ , then  $R[1, j]$  (resp.  $R[2, j]$ ) stores that vertex itself. A zero entry in  $R[1, j]$  and  $R[2, j]$  indicate that  $P_j$  is entirely not visible to  $\pi$ . Note that, if more than one part of  $P_j$  is visible from  $\pi$  at an instant of time, then that information is not stored in the array  $R$ . From now onwards, by the term that a vertex is visible/invisible, we mean that it is visible/invisible to  $\pi$ . The following two structural lemmas are crucial.

**Lemma 2.** *After the processing of  $P_j$ , let some vertices in the chain  $P_k$  ( $k < j$ ) remains visible; the minimum and maximum indices of visible vertices in  $P_k$  be  $f_k$  and  $l_k$ , respectively. Now, if there exists any invisible vertex  $p_\beta \in P_k$  with  $f_k < \beta < l_k$ , then the visibility of  $p_\beta$  can only be obstructed by an edge of the chain  $P_k$  itself.*

*Proof.* If the visibility of  $p_\beta$  is obstructed by some edge of an unvisited polychain, then it is not yet identified. So, we need to consider the case where  $p_\beta$  is obstructed by some edge of a *visited* polychain  $P_\gamma$ , where  $1 < \gamma \leq k$  or  $k < \gamma \leq j$ . In the former case, prior to obstructing  $p_\beta$ , it must have obstructed  $p_{f_k}$ . In the latter case, prior to obstructing  $p_\beta$ , it must have obstructed  $p_{l_k}$ . In both the cases we have contradiction since both  $p_{f_k}$  and  $p_{l_k}$  are visible to  $\pi$ .  $\square$

**Lemma 3.** *During the processing of  $P_j$ , if it is observed that some visible vertices of  $P_k$  ( $k < j$ ) becomes invisible, then all the vertices of  $P_{j-1}, P_{j-2}, \dots, P_{k+1}$  (if any) becomes invisible. Moreover, if a vertex  $p_\alpha \in P_j$  becomes invisible, then all the vertices having index greater than  $\alpha$  in  $P_j$  are invisible.*

Our algorithm consists of two passes. In the first pass, we compute  $f_j$  and  $l_j$  for all the polychains  $P_j$  in anticlockwise order, and set  $R[1, j]$  and  $R[2, j]$  as described above. In the second pass, we consider each  $P_j$  and print the visible portions from  $\pi$  (if any). If  $R[1, j], R[2, j] \neq 0$ , then there exists at least one part of  $P_j$  that is visible from  $\pi$ .

**Pass 1:** In this pass, we process  $P_j$  as in Subsection 2.1. We copy the indices of the vertices of  $P_j$  in array  $S$ , and process them in counterclockwise order. As in Section 2.1, during the execution, the visible portion of  $P_j$  is stored in the stack maintained at the beginning of  $S$ . An index variable  $\ell$  is used as the top pointer of the stack. We also use a variable  $\chi$  that contains the index of the most recently visited polychain which is completely/partially visible at the current instant of time. At the beginning of processing  $P_j$ ,  $\chi$  is initialized with  $j - 1$ .

While processing a vertex  $p \in P_j$ , here also three cases may arise. The processing of different cases are same as in Subsection 2.1 except Case 2 and Case 3.1.

In Case 2, if the last vertex  $p$  of  $P_j$  is not visible (i.e.  $\angle q\pi S[\ell] > \angle q\pi p$ ), then we store  $S[\ell]$  in a temporary variable  $\sigma$ . We set  $R[2, j] = R[1, j + 1] = \sigma$ , and copy  $P_{j+1}$  in  $S$  for the processing.

In Case 3.1, the first task is to put the index of  $p$  in  $R[1, j]$ . Next, we check whether the vertex  $p$  blocks the visibility of some already processed polychains by considering them in clockwise order starting from  $P_\chi$ . By Lemma 3, a polychain  $P_k$  is considered if all the polychains  $\{P_\gamma, k < \gamma < j\}$  become invisible. While considering  $P_k$ , we may have the following three situations. Let the vertex  $p'$  be the clockwise neighbor of the vertex  $p \in P_j$ ,  $R[1, k] = \alpha$  and  $R[2, k] = \beta$ .

**Case 3.1.1 - The edge  $(p', p)$  does not intersect the lines  $[\pi, p_\alpha]$  and  $[\pi, p_\beta]$ :** Here the entire visible portion of the polychain  $P_k$  remains visible, and we need not have to test the other visited polychains  $P_\gamma$ ,  $\gamma < k$ . So, we push  $p$  on to the stack maintained in  $S$ , and consider the vertex next to  $p$  in the polygon  $P$  for processing.

**Case 3.1.2 - The edge  $(p', p)$  of  $P$  intersects the line  $[\pi, p_\beta]$  but does not intersect  $[\pi, p_\alpha]$ :** Here, we replace  $\beta$  by the index of  $p$  in  $R[2, k]$ . Next,  $p$  is pushed in  $S$  and the vertex next to  $p$  is considered for processing.

**Case 3.1.3 - The edge  $(p', p)$  of  $P$  intersects both the lines  $[\pi, p_\alpha]$  and  $[\pi, p_\beta]$ :** Here the entire polychain  $P_k$  becomes invisible. We set  $R[1, k] = R[2, k] = 0$ , and consider the next visible polychain in clockwise order. The process continues until we arrive at either Case 3.1.1 or Case 3.1.2.

At the end of processing  $P_j$  (if special Case 2, as mentioned before, doesn't arise), we put the index of the last vertex of  $P_j$  in  $R[2, j]$ .

**Pass 2:** In this pass, we consider each polychain  $P_j$  for printing its visible portion from  $\pi$ . We start from  $P_1$ , and proceed in counterclockwise order. If

$R[1, j], R[2, j] \neq 0$  for a polychain  $P_j$ , then it is fully/partially visible from  $\pi$ . We compute the leftmost and rightmost points of  $P_j$ , say  $f_j$  and  $l_j$  that are visible from  $\pi$  as follows.

If  $R[1, j]$  points to the leftmost point of  $P_j$ , then  $f_j = R[1, j]$ . Otherwise, we identify the edge  $e$  that is hit by the ray  $\overrightarrow{\pi p_{\alpha'}}$  first, where  $\alpha' = R[1, j]$ . We compute  $\phi$  = point of intersection of  $e$  and  $\overrightarrow{\pi p_{\alpha'}}$ .

Similarly, if  $R[2, j]$  points to the rightmost point of  $P_i$ , then  $l_j = R[2, j]$ . Otherwise, we identify the edge  $e$  that is hit by the ray  $\overrightarrow{\pi p_{\beta'}}$  first, where  $\beta' = R[2, j]$ . We compute  $\psi$  = point of intersection of  $e$  and  $\overrightarrow{\pi p_{\beta'}}$ .

Next we copy the indices of all the vertices of  $P_j$  between  $\phi$  and  $\psi$  along with  $\phi$  and  $\psi$  in the array  $S$  in order. By Lemma 3, if there is any invisible vertex in  $S$ , it is blocked by some edge in  $P_j$ . So, we execute INPLACE\_VISIBILITY on the array  $S$ . At the end of processing  $P_j$ , the content of the stack is printed.

**Theorem 2.** *The algorithm READONLY\_VISIBILITY correctly computes the visibility polygon of  $P$  from the point  $\pi$ . It needs  $O(n)$  time and  $O(\sqrt{n})$  work-space.*

*Proof.* The correctness of the algorithm READONLY\_VISIBILITY follows from the fact that for each polychain  $P_j$ , if its visibility is blocked from any/both side(s), then blocking vertices are correctly computed as mentioned in the correctness proof of the algorithm INPLACE\_VISIBILITY and are stored in the array  $R$ . In Pass 2, the first and last visible vertices of  $P_j$  can be correctly computed from the content of  $R[1, j]$  and  $R[2, j]$ . The correctness of Pass 2 follows from Lemma 2. We now analyze the complexity results of the algorithm.

While executing Pass 1 on the polychain  $P_j$ , copying the polychain  $P_j$  in  $S$  needs  $O(\sqrt{n})$  time. During the processing of Pass 1, each vertex in  $P_j$  is inserted in and deleted from stack at most once. In the entire Pass 1, if a polychain becomes completely invisible, i.e.,  $R[1, j], R[2, j]$  are set to 0, it never becomes visible. Note that, during the processing of each vertex in  $P_j$ , the visibility of at most one polychain is reduced by changing its  $R[1, j]$  field. Thus, the amortized time complexity for processing each vertex in Pass 1 is  $O(1)$ . While executing Pass 2 for each  $P_i$ , we first spend  $O(\sqrt{n})$  time for computing the first and last visible vertices  $\phi$  and  $\psi$ . The copying of the visible portion of  $P_i$  in  $S$ , and executing INPLACE\_VISIBILITY on  $S$  needs another  $O(\sqrt{n})$  time in the worst case. Since we have  $\lceil \sqrt{n} \rceil$  polychains, the result follows.  $\square$

### 3 Weak Visibility Polygon of an edge

Given a polygon  $P$  and a line segment  $\ell = [p, q]$  in  $P$ , the weak visibility polygon of  $\ell$ , denoted by  $WVP(\ell)$ , is a simple polygonal region  $R$  such that each point in  $R$  is visible from at least one point of  $\ell$ . As in the earlier section, here also we will assume that the vertices  $\{p_1, p_2, \dots, p_n\}$  of the polygon  $P$  are given in a read-only array, called  $P$ , in anticlockwise order. We will propose an algorithm for computing the weak-visibility polygon of an edge  $e = (p_i, p_{i+1})$  of  $P$ , denoted by  $WVP(e)$  (see Figure 3).

Before presenting the algorithm for computing  $WVP(e)$ , let us consider first the following two simpler problems.

**LEFTNONVISIBLE( $e, e'$ ):** Given two edges  $e$  and  $e'$  of a polygon  $P$ , compute the portion of  $e'$ , from its left endpoint, which is not weakly-visible from  $e$ .  
**RIGHTNONVISIBLE( $e, e'$ ):** Given two edges  $e$  and  $e'$  of a polygon  $P$ , compute the portion of  $e'$ , from its right endpoint, which is not weakly-visible from  $e$ .  
We will explain the method of solving the problem **LEFTNONVISIBLE( $e, e'$ )**. **RIGHTNONVISIBLE( $e, e'$ )** will be symmetric to that. Once we have the solution of these two problems, we will be able to compute the portion of  $e'$  which is weakly visible from  $e$ . In order to compute  $WVP(e)$ , we need to compute the weak-visible portion of all the edges  $e' \neq e$  of the polygon  $P$ .

### 3.1 LEFTNONVISIBLE( $e, e'$ )

Let  $e = [p_i, p_{i+1}]$  and  $e' = [p_j, p_{j+1}]$ ,  $j > i$ . We first join  $[p_{i+1}, p_{j+1}]$  (say  $L$ ). We will use  $\theta$  and  $\theta'$  to denote the end-points of  $L$  on  $e$  and  $e'$  respectively. At the end of the execution, it returns  $\Phi$  = left non-visible portion of  $e'$  from  $e$ . Initially  $\theta = p_{i+1}$  and  $\theta' = p_{j+1}$ . The algorithm consists of three passes.

**Pass-1:** In this pass, polychain  $\Pi = [p_{j+1}, p_{j+2}, \dots, p_n, p_1, p_2, \dots, p_i]$  is traversed in a counterclockwise manner starting from  $p_{j+1}$ . If a vertex  $p \in \Pi$  is observed which is to the right of  $L$  in its present position, then  $\theta'$  is moved to the point of intersection of the line containing  $e'$  and the line  $(p_{i+1}, p)$ ;  $\theta$  remains fixed at  $p_{i+1}$ . If  $\theta'$  is outside  $e'$ , then  $e'$  is not visible from  $e$ , and the procedure returns  $\Phi = e'$ . Otherwise,  $L$  is defined by  $p_{i+1}$  and a vertex  $p_\tau \in \Pi$  (see Figure 4(a)). A temporary variable  $\tau$  is used to remember  $p_\tau$ .

**Pass-2:** In this pass, we simultaneously traverse  $\Pi$  from  $p_i$  in clockwise order up to the vertex  $p_\tau$ , and  $\Pi'$  from  $p_{i+1}$  to  $p_j$  in counterclockwise order. The method of traversal is explained in *Process-1* and *Process-2*, stated below. We use three index variables  $k, \ell$  and  $m$ . Initially, we set  $k = i + 1, \ell = i$  and  $m = \tau$ .

**Process-1:** We traverse  $\Pi$  in anticlockwise direction using the index variable  $k$ . At each move one of the following events may be observed:

- $k = j$ . Pass 2 stops, and it returns  $\Phi = [p_{j+1}, \theta']$ .
- $k \neq j$  and the edge  $(p_{k-1}, p_k)$  does not intersect  $L$ . Here  $k$  is incremented (mod  $n$ ) to process the next vertex of  $\Pi$ .
- $k \neq j$  and the edge  $(p_{k-1}, p_k)$  intersects  $L$  above  $p_m$ . Here,  $e'$  is not weakly visible to  $e$  (see Figure 4(a)), The procedure returns  $\Phi = e'$ .
- $k \neq j$  and the edge  $(p_{k-1}, p_k)$  intersects  $L$  below  $p_m$ . Here, we update  $L$  with the line joining  $(p_m, p_k)$ , and update  $\theta$  (resp.  $\theta'$ ) by the point of intersection of  $L$  and  $e$  (resp.  $e'$ ) (see Figure 4(b)). Next we execute *Process-2*.

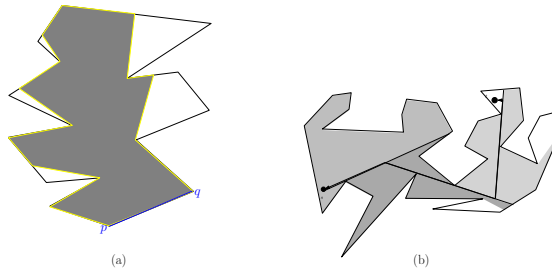


Fig. 3: Demonstration of (a) Weak visibility polygon, (b) minimum link path



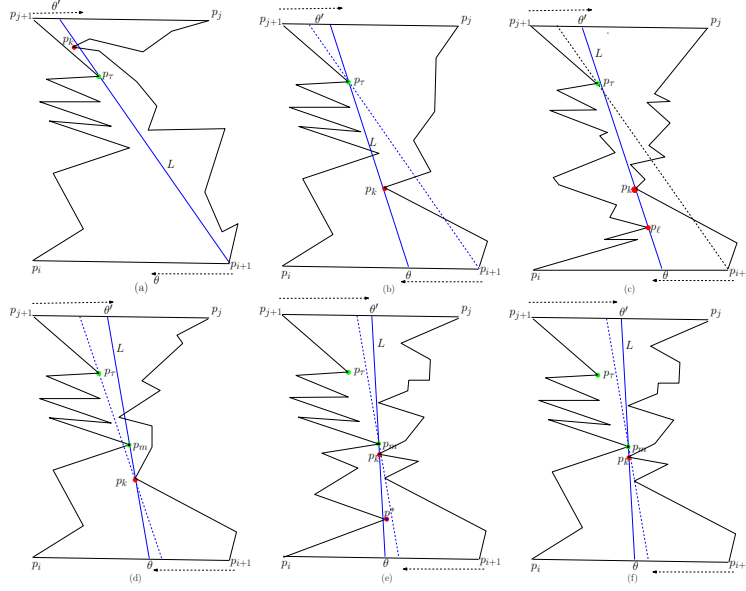


Fig. 4: Algorithm LEFTNONVISIBLE

**Process-2:** We traverse  $\Pi'$  in clockwise direction using the index variable  $\ell$ . At each move the following events may happen:

- the edge  $(p_{\ell+1}, p_\ell)$  does not intersect  $L$ . Here,  $\ell$  is decremented (mod  $n$ ) to process the next vertex of  $\Pi'$ .
- the edge  $(p_{\ell+1}, p_\ell)$  intersects  $L$  below  $p_k$ . Here,  $e'$  is not weakly visible to  $e$  (see Figure 4(c)). The procedure returns  $\Phi = e'$ .
- the edge  $(p_{\ell+1}, p_\ell)$  intersects  $L$  above  $p_k$ . Here, we update the  $L$  by the line joining  $p_\ell$  and  $p_k$ . We also set  $\theta' = \pi$  and  $m = \ell$ , and then switch to executing the *Process-1*.

**Pass-3:** Note that after executing *Process 2*,  $L$  may intersect  $\Pi$  below  $p_k$  (see Figure 4(e)). Thus the weak-visibility of  $e'$  from  $e$  may be lost. So, after the successful finish of Pass-2, we execute Pass-3 to check for possible intersection of  $L$  and the edges of  $\Pi'$ . If such an intersection is observed,  $e'$  is not weakly visible from  $e$ ; otherwise, the final position of  $\theta'$  determines the non-visible portion  $\Phi = [p_{j+1}, \theta']$  of  $e'$  from  $e$  (see Figure 4(f)).

**Lemma 4.** *Algorithm LEFTNONVISIBLE correctly computes the invisible portion of  $e'$  from its left end-point, and it needs  $O(n)$  time and  $O(1)$  space.*

*Proof.* Initially, we draw the line segment  $L = [p_{i+1}, p_{j+1}]$ . Now, three cases may arise: (i) the entire segment  $L$  lies inside the polygon, (ii) the entire segment  $L$  lies outside the polygon, and (iii)  $L$  spans both inside and outside the polygon. In case (i),  $e'$  starting from  $p_{j+1}$  is visible. In other two cases, there may exist situations where  $e'$  is/is not weakly visible from  $e$ . We now show that in both cases our algorithm correctly computes the left non-visible portion of  $e'$ .

In Pass-1, we compute the invisible portion of  $e'$  from left by traversing the left chain  $\Pi = [p_{j+1} \sim p_i]$  (assuming the right chain is absent). At each step of traversal, if the current edge obstructs  $L$ , then  $L$  is shifted towards right along  $e'$ , and in the further scanning of the chain  $L$  is never shifted towards left.

Now, we need to consider the right chain  $\Pi'$ . The visibility may be blocked by some vertex of  $\Pi'$  at the current position of  $L$ . So, we traverse the right chain from  $p_{i+1}$  to  $p_j$ . As soon as a blocking is observed,  $L$  is modified as mentioned in the algorithm. However, in the new position of  $L$ , a portion of it becomes closer to the left chain  $\Pi$ ; thus, it may again be obstructed by  $\Pi$ . So, we need to traverse  $\Pi$ . This alternating process may continue until  $p_j$  is reached along the right chain. It needs to be mentioned that during this traversal, we have not noticed whether  $L$  is obstructed by some edge which is already visited. So, finally we execute Pass-3 to check this.

The time complexity follows from the fact that  $\Pi$  is traversed three times and  $\Pi'$  is traversed once only. During the traversal of  $\Pi$  (resp.  $\Pi'$ ) its each vertex is visited at most once.  $\square$

**Theorem 3.** *The weak visibility polygon of an edge of  $P$  can be computed in  $O(n^2)$  time using  $O(1)$  extra space, where the vertices of  $P$  are given in a read-only array.*

## 4 Minimum link path between a pair of points

Given a polygon  $P$  and a pair of points  $s$  and  $t$ , the *minimum link path* between  $s$  and  $t$ , denoted by  $MLP(s, t)$ , is a polygonal chain from  $s$  to  $t$  where the number of edges in the chain is minimum among all other polygonal chains connecting  $s$  and  $t$  (Figure 3(b)). We propose an algorithm for computing  $MLP(s, t)$  assuming that the vertices of  $P$  are given in a readonly array in anticlockwise order.

A classic way to compute  $MLP(s, t)$  is as follows: (i) Compute the visibility polygon of  $s$ , called  $Q$ . If  $t \in Q$ , then  $s$  and  $t$  are straight-line visible. Otherwise (ii) identify the edge  $\chi$  of  $Q$  such that the sub-polygons of  $P$  lying on one side of  $\chi$  contains  $s$  and that on the other side contains  $t$ . Now (iii) compute the weak visibility polygon of  $\chi$  in the sub-polygon containing  $t$ . If again it contains  $t$ , then the process stops; otherwise iterate steps (ii) and (iii). The time complexity is  $O(kn^2)$  and it needs  $O(1)$  extra space (see Theorem 3), where  $k$  is the number of segments in  $MLP(s, t)$ .

We now describe an algorithm for this problem that runs in  $O(kn)$  time and  $O(1)$  work-space. It also executes  $k$  iterations to report the  $k$  segments of the path. We use a variable  $\pi$  to store the point such that the path from  $s$  to  $\pi$  is already reported. Initially  $\pi$  stores  $s$ . In each iteration we execute the procedures  $LEFTNONVISIBLE(e, e')$  described in Section 4. It is tailored such that it can work even if  $e$  and/or  $e'$  is/are point(s) inside  $P$ , and returns two parameters  $\Phi$  and  $L$ , where  $\Phi$  is the left non-visible portion of  $e$  from  $e' = t$  and  $L$  is a line such that nothing to the left of  $L$  is visible to  $e' = t$ . If  $e$  is a point, then we draw a horizontal line segment  $[a, b]$  inside  $P$  that passes through  $e$  and the points  $a, b$  lying on the boundary of  $P$ . Similarly, the line segment  $[c, d]$  is defined for

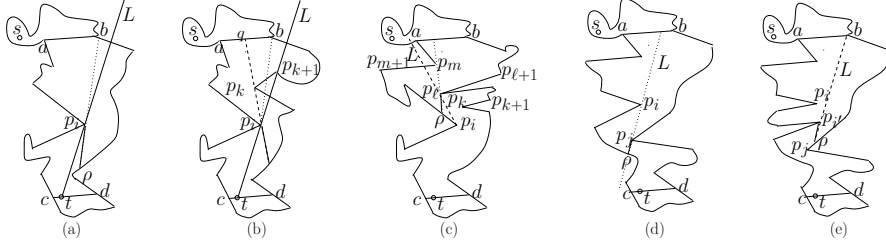


Fig. 5: Computation of minimum link path

$e'$  if it is a point. Initially,  $e$  and  $e'$  are the points  $s$  and  $t$ , respectively. Here  $\Pi$  is defined as an anticlockwise polychain from  $a$  to  $c$  and  $\Pi'$  is a clockwise polychain from  $b$  to  $d$ . From next iteration onwards  $e = [a, b]$  will be an edge of the corresponding polygon, but  $e'$  remains equal to  $t$  in all the iterations.

At each step, if  $\Phi \neq e$  and one of the end-points of  $L$  is equal to  $t$ , then  $t$  is visible from some point of  $e$ . We choose the left-most point  $q$  of  $\Phi$ , and report the last two edges  $[\pi, q]$  and  $[q, t]$  of the MINIMUM\_LINK\_PATH.

If  $\Phi = e$ , then no part of  $e$  is visible from  $t$ ; We execute the procedure Compute-First-Link( $\Phi, L$ ), stated below to compute an edge  $\Psi$  of the weak-visibility polygon of  $e$  such that  $s$  and  $t$  are in two different sides of  $\Psi$ .

**Observation 2** (i) Every point of  $\Psi$  is weakly visible from  $e$  and no point in the proper interior of the sub-polygon of  $P$  containing  $t$  is visible from  $e$ .  
(ii) Extension of  $\Psi$  intersects  $e$  (at a point, say  $q$ ).

We report the link  $[\pi, q]$ ; reset  $\pi = q$ , and execute the next iteration with  $e = \Psi$ .

#### 4.1 Compute-First-Link

Here we need to handle the following two cases: (i)  $L$  does not intersect the edge  $e$  but contains the point  $t$ , and (ii)  $L$  does not contain the point  $t$ .

**Case (i)** Let  $L$  be defined by a vertex  $p_i \in \Pi$  and  $t$  (see Figure 5). We compute the right non-visible portion of  $e = [a, b]$  from the point  $p_i$ . We redefine  $L = [b, p_i]$  and alternately visit the vertices of the polychain  $\Pi'$  from  $d$  to  $b$  in anticlockwise order and the polychain  $\Pi$  from  $p_i$  to  $a$  in clockwise order using the procedures **Process-1** and **Process-2** stated below. Note that,  $[c, d]$  is a chord of  $P$  containing  $t$ , as stated earlier.

**Process-1:**

- If no edge of  $\Pi'$  intersects  $L$  above  $p_i$  then it returns  $\Psi = [p_i, \rho]$ , where  $\rho$  is the point of intersection of  $L$  and  $\Pi'$  (see Figure 5(a)).
- If an edge  $(p_k, p_{k+1})$  of  $\Pi'$  intersects  $L$ , then  $L$  is modified by  $[p_i, p_k]$ , and the end-point of  $L$  on  $e$  moves towards  $a$ . For each encounter of an edge of  $\Pi'$  intersecting  $L$ ,  $L$  is modified accordingly. The traversal continues along  $\Pi'$  until  $b$  is reached (Figure 5(b)) or an edge  $(p_\ell, p_{\ell+1}) \in \Pi'$  is encountered such that the redefined  $L$  does not intersect  $e$  (Figure 5(c)). Now **Process-2** is invoked.

**Process-2:**

We redefine  $L = [p_\ell, a]$  and start traversing  $\Pi$  from  $p_i$  towards  $a$  to compute the left non-visible portion of  $e$  from the point  $p_\ell$ . As in **Process-1**, for each encounter of an edge  $(p_m, p_{m+1}) \in \Pi$  intersecting  $L$ , the end-point of  $L$  on  $e$  moves towards  $b$ . Finally

- If  $a$  is reached and  $L$  does not leave  $e$ , then report  $\Psi = [p_{m+1}, \rho]$ , where  $\rho$  is the point of intersection of  $L$  and the polychain  $\Pi'$  (Figure 5(c)).
- If the end-point of  $L$  on  $e$  goes beyond  $b$  for an edge  $(p_m, p_{m+1}) \in \Pi$ , then  $L$  is redefined as  $[p_{m+1}, b]$  and **Process-1** is invoked.

**Case (ii)** Let  $L$  be defined by two vertices  $p_i \in \Pi$  and  $p_j \in \Pi'$ . As in the algorithm LEFTNONVISIBLE here also we traverse from  $c$  to  $p_i$  along  $\Pi$  to get a vertex that intersects  $[p_j, p_i]$ . If no such vertex is found and  $L$  intersects  $\Pi$  at a point  $\rho$ , then  $\Psi = [\rho, p_j]$  is returned (see Figure 5(d)). If such a vertex  $p_{i'}$  is found, then  $L$  is redefined by  $p_j$  and  $p_{i'}$ , and the traversal from  $p_j$  starts to get a vertex in  $\Pi'$  that intersects  $[p_j, p_{i'}]$  (see Figure 5(e)). This type alternate traversal in  $\Pi$  and  $\Pi'$  ultimately defines a line  $L$  by two vertices  $p_\alpha \in \Pi$  and  $p_\beta \in \Pi'$  such that there exists no edge  $(p_k, p_{k+1})$  in  $\Pi$  with  $k > i$  that intersects  $L$ . Now if  $L$  does not intersect  $e$ , the situation is similar to Case (i). Otherwise, the traversal along  $\Pi'$  continues from  $p_\beta$  until it is obstructed by a vertex in  $\Pi$ . Then the traversal starts from  $p_\alpha \in \Pi$ . This type of alternate traversal in  $\Pi$  and  $\Pi'$  continues as in Case (i) to have a chord of  $P$  defined by its two vertices.

**Theorem 4.** *The proposed algorithm correctly computes the minimum link path between  $s$  and  $t$  in time  $O(kn)$ , where  $P$  is given in a read-only array, and  $k$  is the size of the output.*

*Proof.* The above scheme reports an edge  $\eta$  of the visibility polygon of  $e$  such that  $s$  and  $t$  are in different sides of it. The minimality of the path length follows from the fact that  $\eta$  satisfies Observation 2. The above scheme needs  $O(n)$  time since it needs visiting all the vertices of  $\Pi$  and  $\Pi'$  at most a constant number of times. Since the length of the minimum link path is  $k$ , the result follows.

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## Appendix

---

### Algorithm 1: INPLACE\_VISIBILITY

---

**Input:** A simple polygon in the array  $P[]$  and a point  $\pi$  inside the polygon  
**Output:** Visibility polygon of  $\pi$

```

1 Draw a horizontal line  $\vec{H}$  from  $\pi$  towards right and find the first intersection point  $q$  with
  the polygon  $P$ . Let  $q \in$  the edge  $(p_\theta, p_{\theta+1})$ .;
2 Initialize:  $k = l = \theta + 1$ ;  $i = \theta + 2$ ;  $previous\_vertex = P[\theta + 1]$ ;
3 while  $i \neq \theta$  do
4   (* Case 1: *);
5   if  $\angle q\pi P[l] < \angle q\pi P[i]$  then
6     Push_and_Proceed( $i, l$ );
7   (* Case 2: *);
8   else if  $\angle q\pi P[l] > \angle q\pi P[i]$  and the polygon makes a right turn at  $P[i]$  then
9     while  $(\angle q\pi P[l] > \angle q\pi P[i]) \wedge (i \neq \theta)$  do
10      Only_Proceed( $i$ )
11      if  $\angle q\pi P[l] < \angle q\pi P[i]$  then
12         $\phi =$  intersection point of edge  $(p_{i-1}, p_i)$  and the half-line  $[\pi, P[l]]$ ;
13        Push_Phi( $\phi, l$ );
14        PUSH_and_PROCEED( $i, l$ );
15   (* Case 3: *);
16   else
17     while  $!(Case3.1 \vee Case3.2 \vee Case3.3)$  do
18        $\ell = l - 1$ ;
19       (* Case 3.1: *);
20       if  $\ell < k$  then
21          $k = i - 1$ ;  $P[k] = q$ ;  $\ell = i$ ;
22         Only_Proceed( $i$ )
23       (* Case 3.2: *);
24       else if  $\angle q\pi P[l] < \angle q\pi P[i]$  then
25          $\phi =$  intersection point of edge  $(P[l], P[l + 1])$  and the half-line  $[\pi, P[i]]$ ;
26         Push_Phi( $\phi, l$ );
27         PUSH_and_PROCEED( $i, l$ );
28       (* Case 3.3:  $(\angle q\pi P[l] > \angle q\pi P[i]) \wedge ([P[i - 1], P[i]] \text{ and } [P[l], P[l + 1]] \text{ properly intersect})$  *);
29       else
30         while  $!((\angle q\pi P[l] < \angle q\pi P[i]) \wedge (i = \theta))$  do
31           Only_Proceed( $i$ )
32           if  $i \neq \theta$  then
33              $\phi =$  intersection point of the edge  $(previous\_vertex, P[i])$  and the half-line
               $[\pi, P[l]]$ ;
34             Push_Phi( $\phi, l$ );
35             PUSH_and_PROCEED( $i, l$ )
36 Report  $P[k \dots \ell]$  of the array;
37 end.
38
39 Procedure PUSH_and_PROCEED( $i, \ell$ );
40  $\ell = (\ell + 1) \bmod n$ ;  $previous\_vertex = P[i]$ ;
41 swap( $P[i], P[\ell]$ );  $i = (i + 1) \bmod n$ ;
42 end.
43
44 Procedure Only_Proceed( $i$ );
45  $previous\_vertex = P[i]$ ;  $i = (i + 1) \bmod n$ ;
46 end.
47
48 Procedure Push_Phi( $\phi, \ell$ );
49  $\ell = (\ell + 1) \bmod n$ ;  $P[\ell] = \phi$ ;
50 end.
```

---

---

**Algorithm 2: Modified-Case-3.1**

---

**WorkSpace:** An array  $R$  of size  $2 \times \lceil \sqrt{n} \rceil$ , and array  $S$  of size  $\lfloor \sqrt{n} \rfloor$

- 1  $R[1, j]$  and  $R[2, j]$  contain the blocking vertices of  $P_j$  from its two sides;
- 2 While processing  $P_j$ , it is copied in  $S$  and processed there. The array  $P$  remains unaltered;
- 3 Variables used;
- 4  $\ell$ : top pointer of stack maintained at the beginning of  $S$ ;
- 5  $\chi$ : index of the poly-chain of maximum index that is totally/partially visible;
- 6  $i$ : index of the current vertex in  $S$ ;
- 7 (\* Modified Case 3.1: Here  $\ell$  is reached to 0 \*);
- 8 **while**  $(R[1, \chi] = 0 \wedge R[2, \chi] = 0) \vee ((\text{previous\_vertex}, S[i]) \text{ intersects both the line } [\pi, p_\alpha) \text{ and } [\pi, p_\beta)) \vee (\chi \neq 0)$  **do**
- 9     **if**  $(\text{previous\_vertex}, S[i]) \text{ intersects both the line } [\pi, p_\alpha) \text{ and } [\pi, p_\beta)$  **then**
- 10          $(R[1, \chi] = 0; R[2, \chi] = 0;$
- 11          $\chi = \chi - 1;$
- 12 **if**  $\chi \neq 0$  **then**
- 13     **if**  $(\text{previous\_vertex}, S[i]) \text{ intersects none of the line } [\pi, p_\alpha) \text{ and } [\pi, p_\beta)$  **then**
- 14          $\text{previous\_vertex} = S[i]; \ell = \ell + 1; S[\ell] = S[i]; i = i + 1;$
- 15     **else**
- 16          $R[2, \chi] = i;$
- 17          $\text{previous\_vertex} = S[i]; \ell = \ell + 1; S[\ell] = S[i]; i = i + 1;$

---

---

**Algorithm 3: Pass 2 of Algorithm READONLY\_VISIBILITY**

---

- 1  $j = 1;$
- 2 **while**  $j \neq \text{No\_of\_Partition}$  **do**
- 3     **if**  $R[1, j] \neq 0 \wedge R[2, j] \neq 0$  **then**
- 4          $\alpha = R[1, j]; \beta = R[2, j];$
- 5         **if**  $P[\alpha] \notin P_j$  **then**
- 6             Find the edge  $e$  of  $P_j$  which is intersected first by the line  $[\pi, P[\alpha]);$
- 7              $\phi = \text{intersection of } e \text{ and } [\pi, P[\alpha])$
- 8         **else**
- 9              $\phi = P[\alpha]$
- 10         **if**  $P[\beta] \notin P_j$  **then**
- 11             Find the edge  $e$  of  $P_j$  which is intersected first by the line  $[\pi, P[\beta]);$
- 12              $\psi = \text{intersection of } e \text{ and } [\pi, P[\beta])$
- 13         **else**
- 14              $\psi = P[\beta]$
- 15         Copy  $\phi$  to  $\psi$  of the polychain  $S$ ;
- 16         Execute Algorithm INPLACE\_VISIBILITY on  $S$  to report the visible portion
- 17      $j = j + 1$

---

---

**Algorithm 4:** LEFTNONVISIBLE( $v_i v_{i+1}, v_j v_{j+1}$ )

---

**Input:** Two edges  $p_i p_{i+1}$  and  $p_j p_{j+1}$  of a simple polygon  $P$   
**Output:** The left portion of  $p_j p_{j+1}$  which is not weakly-visible from  $p_i p_{i+1}$

```
1  PASS1();
2   $r = PASS2()$ ;
3  if  $r = -1$  then
4  |   Report nothing is visible;
5  else
6  |    $PASS3()$ ;
7  end.
8
9  Procedure  $PASS1()$ ;
10  $L = (p_{j+1}, p_{i+1})$ ;
11  $t = (j + 2) \bmod n$ ;
12 while  $t \neq i$  do
13 |   if  $p_t$  is to the right of  $L$  then
14 |   |    $UPDATELINE(p_t, p_{i+1})$ ;
15 |   |    $\tau = t$ ;
16 |    $t = (t + 1) \bmod n$ ;
17 end.
18
19 Procedure  $PASS2()$ ;
20  $k = i + 1$ ;  $\ell = i$ ;  $m = \tau$ ;
21  $r = 1$ ;
22 while  $r \neq 0 \wedge r \neq -1$  do
23 |   if  $r = 1$  then
24 |   |    $Process1()$ ;
25 |   else if  $r = 2$  then
26 |   |    $Process2()$ ;
27 Return  $r$ ;
28 end.
29
30 Procedure  $Process1()$ 
31 if  $k = j$  then
32 |    $r = 0$ ;
33 else
34 |   if  $(p_{k-1} p_k)$  does not intersect line  $L$  then
35 |   |    $k = (k + 1) \bmod n$ ;
36 |   |    $r = 1$ ;
37 |   else if  $(p_{k-1} p_k)$  intersects below  $p_m$  then
38 |   |    $UPDATELINE(p_m, p_k)$ ;
39 |   |    $r = 2$ ;
40 |   else
41 |   |    $r = -1$ ;
42 end.
43
44 Procedure  $Process2()$ 
45 if  $l > \tau$  then
46 |   if  $p_{\ell+1} p_\ell$  does not intersect  $L$  then
47 |   |    $\ell = (\ell - 1) \bmod n$ ;
48 |   |    $r = 2$ ;
49 |   else if  $p_{\ell+1} p_\ell$  intersects  $L$  below  $p_k$  then
50 |   |    $r = -1$ ;
51 |   else
52 |   |    $UPDATELINE(p_l, p_k)$ ;
53 |   |    $m = \ell$ ;
54 |   |    $r = 1$ ;
55 else
56 |    $r = 0$ ;
57 end.
58
59 Procedure  $PASS3()$ 
60  $t = (j + 2) \bmod n$ ;
61 while  $(t \neq i) \bigwedge (r \neq -1)$  do
62 |   if  $p_t$  is to the right of  $L$  then
63 |   |    $r = -1$ ;
64 |    $t = (t + 1) \bmod n$ ;
65  $t = (i + 2) \bmod n$ ;
66 while  $(t \neq j) \bigwedge (r \neq -1)$  do
67 |   if  $p_t$  is to the left of  $L$  then
68 |   |    $r = -1$ ;
69 |    $t = (t + 1) \bmod n$ ;
70 if  $r = -1$  then
71 |   Report nothing is visible;
72 else
73 |   Report  $(p_{j+1}, \theta')$  is not visible;
74 end.
```

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